

Classification of symmetry fractionalization in gapped \mathbb{Z}_2 spin liquids

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In quantum spin liquids, fractional spinon excitations carry half-integer spins and other fractional quantum numbers of lattice and time-reversal symmetries. Different patterns of symmetry fractionalization distinguish different spin liquid phases. In this work, we derive a general constraint on the symmetry fractionalization of spinons in a gapped spin liquid, realized in a system with an odd number of spin- $\frac{1}{2}$ per unit cell. In particular, when applied to kagome/triangular lattices, we obtain a complete classification of symmetric gapped \mathbb{Z}_2 spin liquids.

I. INTRODUCTION

Gapped spin liquids [1, 2] are strongly-correlated quantum states of spin systems that do not exhibit any Landau-type symmetry-breaking orders [3], but instead are characterized by the so-called topological order [4–7]. The existence of the topological order manifests as the emergence of quasiparticle excitations with anyonic braiding and exchange statistics. Although no symmetries are broken, the nontrivial interplay between the symmetries and the topological degrees of freedom further enriches the concept of topological orders, leading to the notion of symmetry-enriched topological (SET) order [8]. In SET phases, global symmetries can have nontrivial actions on the quasiparticle excitations, which are otherwise not allowed on non-fractionalized local excitations (i.e. magnons). In particular, quasiparticle excitations can carry fractionalized symmetry quantum numbers, a phenomena known as symmetry fractionalization [9].

An ubiquitous pattern of symmetry fractionalization in quantum spin liquids is that there exists a quasiparticle excitation, dubbed the “spinon” [4, 7], which carries a half-integer spin quantum number [10]. Besides the spin-rotational symmetry, quantum spin liquids often have time-reversal symmetry as well as crystalline symmetries of the underlying lattice. Correspondingly, the spinon can also have fractionalized quantum numbers of the space-time symmetries. This was first discussed in the framework of parton constructions of spin liquid state, known as the projective symmetry group (PSG) of the parton mean-field ansatz [11]. In this work, we will use spinon PSG to collectively refer to space-time symmetry fractionalization on spinons. Together with other fractionalized symmetry quantum numbers, PSG classifies symmetric quantum spin liquids that can be realized on a certain type of lattice.

Classification of symmetry fractionalization is not only of central importance to the understanding of SET phases, but also essential to the studies of quantum spin liquids. Spin liquids with different patterns of symmetry fractionalization belong to different phases, and thus are represented by different symmetric variational wavefunctions [11–15]. Hence, a complete classification exhausts all possible universality classes of topological spin

liquids greatly facilitates the numerical studies of such exotic states. Furthermore, symmetry fractionalization may lead to interesting physical signatures [11, 16–19], which can then be detected both numerically and experimentally. Therefore, symmetry fractionalization provides an important set of features one can use to study the seemingly featureless quantum spin liquid states.

Viewing symmetric gapped quantum spin liquids as SET states, it has been realized that PSGs of different parton constructions can be unified [20], and together they classify a subset of possible SET states [16]. In addition to the spinon PSG, the SET states are also characterized by fractionalized quantum numbers of the vison excitations, which we will briefly refer to as the vison PSG, and possibly an additional layer of symmetry-protected topological (SPT) state [21] (in this work, we will ignore this further distinction, i.e. we regard two SET states related by stacking a SPT layer as the same state). Naively, taking vison PSGs into account would significantly enlarge the classification table [16]. However, it turns out that many SETs where both spinons and visons exhibit nontrivial symmetry fractionalization are anomalous [22–26, 47]: they can not arise in truly two-dimensional lattice systems, and can only be realized on the surface of a three-dimensional SPT state. For on-site unitary symmetries, whether a particular symmetry fractionalization pattern is anomalous or not can be determined mathematically [27, 28]. Unfortunately, such a mathematical framework does not yet exist for space-time symmetries. Instead, in this work, with the assumption of a background spinon charge per unit cell, we are able to obtain a set of nontrivial constraints on non-anomalous 2D spin liquids.

More specifically, we derive constraints on spinon’s symmetry fractionalization in gapped \mathbb{Z}_2 spin liquids with a background spinon charge per unit cell, using the triangular/kagome lattice (these two lattices have the same crystalline symmetry group) as the main example. We show that in these systems the spinon PSG must match the projective representation of the physical degrees of freedom in a unit cell. Similar constraints are introduced in Cheng *et al.* [29] for onsite symmetries. In this work, we show how to generalize this type of constraints to include point-group symmetries. To

gether with the constraints on vison's symmetry fractionalization derived in Ref. [26], we have for the first time obtained a full classification (up to stacking SPT layers) of symmetric gapped \mathbb{Z}_2 spin liquids on a triangular/kagome lattice in two dimensions.

We will focus on possible gapped \mathbb{Z}_2 spin liquids in spin- $\frac{1}{2}$ antiferromagnetic(AF) Heisenberg models on the triangular and kagome lattices, because they have become the most promising candidate systems to realize these exotic phases of quantum spins. Numerical simulations using the density matrix renormalization group (DMRG) method [30–34] have provided evidences that the nearest-neighbor AF Heisenberg model on the kagome lattice has a gapped \mathbb{Z}_2 spin liquid ground state. Moreover, recent experiments [35, 36] on the kagome lattice material herbertsmithite suggest that there is a spin gap. Although neither the numerical nor the experimental evidences are spotless [37, 38], we still use the kagome lattice as the main example to demonstrate our method, which can be readily generalized to other lattices.

The rest of the paper is organized as follows: In Sec. II, we briefly review the concept of symmetry fractionalization and explain its mathematical description, which will be used throughout this paper. In Sec. III, we derive the key result of this work, which is a constraint that relates the symmetry fractionalization class of the spinon to the projective symmetry representation of the physical degrees of freedom in a unit cell. In Sec. IV, we derive the relation between anyonic flux density created by a symmetry operation, and the fractionalization of commutation relations between this symmetry operation and translations. This relation is needed in applying the constraint obtained in Sec. III. In Sec. V, this constraint is applied to the triangular/kagome lattices to obtain a full classification of symmetric gapped \mathbb{Z}_2 spin liquid states on such lattices.

II. SYMMETRY FRACTIONALIZATION IN \mathbb{Z}_2 SPIN LIQUIDS

In this section, we briefly review the physical phenomena of symmetry fractionalization in the context of gapped 2D \mathbb{Z}_2 spin liquids [16, 28]. We are interested in symmetric \mathbb{Z}_2 spin liquids realized in lattice models of spin-1/2 magnets. The largest symmetry group of such lattice models is $\mathcal{G} = G \times G_s$ with G_s being the space group. The “on-site” part of the symmetry group is $G = \text{SO}(3) \times \mathbb{Z}_2^T$, where $\text{SO}(3)$ denotes the spin-rotational symmetry group, and \mathbb{Z}_2^T denotes the time-reversal symmetry group. For concreteness, we define the space group operations to only change the positions of the lattice spins: namely, for $\mathbf{g} \in G_s$, the symmetry operation $R_{\mathbf{g}}$ acts on the spin at \mathbf{r} as $R_{\mathbf{g}} \mathbf{S}_{\mathbf{r}} R_{\mathbf{g}}^{-1} = \mathbf{S}_{\mathbf{g}(\mathbf{r})}$, without any actions on the internal degrees of freedom.

The concept of symmetry fractionalization refers to the phenomena that in a 2D topologically ordered state, non-trivial anyon excitations can carry fractionalized sym-

metry quantum numbers, and more generally, projective representations of the symmetry group G . In the contrary, trivial local excitations must carry “integer” quantum numbers, or linear representations of the symmetry group. Under the assumption of symmetry localization [9], global symmetry transformations acting on a state with multiple well-separated anyons a_1, a_2, \dots should factor into unitaries U^{a_j} localized near the positions of the anyons (for space group transformations, one also has to move the locations of the anyons).

Although the global state must form a linear representation of the symmetry group, the local unitaries can obey the group multiplication law projectively:

$$U^a(\mathbf{g})U^a(\mathbf{h}) = \omega_a(\mathbf{g}, \mathbf{h})U^a(\mathbf{gh}), \quad (1)$$

where each pair of group elements \mathbf{g}, \mathbf{h} is associated with a $U(1)$ phase $\omega_a(\mathbf{g}, \mathbf{h})$. In addition, the projective phases must satisfy the fusion rules of anyons: if the anyon type c appears in the fusion of anyons a and b , then $\omega_a \omega_b = \omega_c$. A well-known result from the algebraic theory of anyons [28] says that all such phase factors consistent with fusion rules can be expressed using mutual braidings with an Abelian anyon:

$$\omega_a(\mathbf{g}, \mathbf{h}) = M_{a, \mathbf{w}(\mathbf{g}, \mathbf{h})}, \quad \mathbf{w}(\mathbf{g}, \mathbf{h}) \in \mathcal{A}. \quad (2)$$

Here \mathcal{A} is the group of Abelian anyons. Therefore equivalence classes of the phase factors $\omega_a(\mathbf{g}, \mathbf{h})$ consistent with both the fusion rule and the associativity law of \mathcal{G} are parametrized by the so-called fractionalization classes $[\mathbf{w}]$, classified by the second group cohomology $\mathcal{H}^2[\mathcal{G}, \mathcal{A}]$ [27, 28, 39].

Notice that we have provided two equivalent descriptions of symmetry fractionalization, in terms of the phase factors $\omega_a(\mathbf{g}, \mathbf{h})$ or the anyon-valued fractionalization class $\mathbf{w}(\mathbf{g}, \mathbf{h})$. We shall use them interchangeably at different places in this paper, depending on convenience.

Now we make these abstract definitions more explicit for \mathbb{Z}_2 spin liquids. There are four types of anyons: the trivial excitations $\mathbb{1}$, the bosonic spinon e , the bosonic vison m and the fermionic spinon ϵ . Under fusion they form a group $\mathcal{A} = \mathbb{Z}_2 \times \mathbb{Z}_2$, generated by e and m . In terms of the fractionalization class $[\mathbf{w}]$, symmetry fractionalization is classified by $\mathcal{H}^2[\mathcal{G}, \mathcal{A}] = \mathcal{H}^2[\mathcal{G}, \mathbb{Z}_2 \times \mathbb{Z}_2] = \mathcal{H}^2[\mathcal{G}, \mathbb{Z}_2]^2$. In terms of the phase factors $\omega_a(\mathbf{g}, \mathbf{h})$, the \mathbb{Z}_2 fusion rules $e \times e = m \times m = \epsilon \times \epsilon = \mathbb{1}$ constraint them to be ± 1 . Therefore, for each type of anyon a the phase factors ω_a are classified by $\mathcal{H}^2[\mathcal{G}, \mathbb{Z}_2]$. Furthermore, the fusion rule $e \times m = \epsilon$ implies $\omega_e = \omega_e \omega_m$. So the independent phase factors are ω_e and ω_m , as expected from the group cohomology classification. [40]

We end this section with two familiar examples of symmetry fractionalization. First, consider the time-reversal symmetry. In most QSLs, both the bosonic spinon and the fermionic spinon are Kramers doublets with $T^2 = -1$ (or $\omega_e(T, T) = -1$), while the vison has $T^2 = +1$ ($\omega_m(T, T) = 1$). The corresponding anyon-valued fractionalization class is $\mathbf{w}(T, T) = m$. Our second example,

which is crucial to our discussions, is the fractionalization of the translation symmetries generated by T_1 and T_2 . As we will see later, QSLs considered in this work all have fractional commutation relation between the two translations for visons: $T_1 T_2 = -T_2 T_1$ (notice that the translation symmetry group is $\mathbb{Z} \times \mathbb{Z}$). In particular, here we consider spin liquids where both the vison m and the fermionic spinon ϵ carries $T_1 T_2 = -T_2 T_1$, and the bosonic spinon e carries $T_1 T_2 = T_2 T_1$. To formally capture this kind of fractionalized commutation relation, it is convenient to define the following symbols $\beta_a(\mathbf{g}, \mathbf{h})$ and $\mathbf{b}(\mathbf{g}, \mathbf{h})$:

$$\beta_a(\mathbf{g}, \mathbf{h}) = \frac{\omega_a(\mathbf{g}, \mathbf{h})}{\omega_a(\mathbf{h}, \mathbf{g})}, \quad \mathbf{b}(\mathbf{g}, \mathbf{h}) = \mathbf{w}(\mathbf{g}, \mathbf{h})\overline{\mathbf{w}(\mathbf{h}, \mathbf{g})}. \quad (3)$$

This definition only applies to two group elements that commute with each other (we will see an example of a generalized version of the “commutator” in Eq. (24) between non-commuting symmetry operations). Using these notations, the fractionalization of translation symmetry is described by $\beta_e(T_1, T_2) = +1$ and $\beta_m(T_1, T_2) = -1$, or equivalently by $\mathbf{b}(T_1, T_2) = e$.

III. MICROSCOPIC CONSTRAINTS ON SPINON’S SYMMETRY FRACTIONALIZATION

In this section, we describe constraints on symmetry fractionalization of spinons in a gapped \mathbb{Z}_2 spin liquid with a net spinon charge per unit cell. We first briefly review the results of Refs. [29 and 41], which relate fractional quantum numbers of spinons to the symmetry representation of the physical degrees of freedom per unit cell, for on-site symmetry operations. Then we generalize this relation to point-group symmetry operations.

We focus on systems with an odd number of spin- $\frac{1}{2}$ per unit cell, which include most of the candidate spin liquid systems studied numerically and experimentally. The Lieb-Schultz-Mattis-Oshikawa-Hastings theorem [42–45] guarantees that a gapped ground state in such systems preserving translation and spin-rotational symmetries must be topological ordered. Recently, Zaletel and Vishwanath [41] and Cheng *et al.* [29] showed that this conclusion can be further refined: symmetry fractionalization in the topological phase is highly constrained by symmetry properties of microscopic degrees of freedom. For example, with $\text{SO}(3)$ symmetry and an odd number of spin-1/2’s per unit cell, it was established [29, 41] that there must be a background anyon charge b in each unit cell that carries spin-1/2. Therefore, b has to a spinon. The physical meaning of the background anyon charge b is that when an anyon a is adiabatically transported around a unit cell, a Berry phase M_{ab} is accumulated. In other words, the translation symmetry is fractionalized: $T_x^{(a)} T_y^{(a)} = M_{ab} T_y^{(a)} T_x^{(a)}$, or equivalently, $\mathbf{w}(T_x, T_y) = b \neq 1$. For \mathbb{Z}_2 QSL, without losing generality we assume that the e anyon and the ϵ anyon carry a half-integer spin [46] So we can have $b = e, \epsilon$. The discussion in this paper generally applies to both cases.

The existence of a nontrivial background spinon charge puts further constraints on the symmetry fractionalization of the spinon. Ref. [29] shows that for an onsite symmetry group G , such constraints relate the fractionalization class to the symmetry quantum numbers of the physical degrees of freedom in the unit cell. Formally, the microscopic degree of freedom in the unit cell can form a projective representation of the symmetry group, labeled by $[\nu] \in \mathcal{H}^2[G, \text{U}(1)]$. The following relation is derived in Ref. [29]:

$$[\nu] = [\omega_b] \cdot [\tau], \quad (4)$$

where $[\tau]$ is a “twist factor” determined by certain symmetry fractionalization class known as the “anyonic spin-orbit coupling (SOC)”, which describes the fractionalization of commutation relations between translations and other symmetries. The precise form of this additional factor will be given later in this section, and in Sec. V, we shall see that for spin liquids on the kagome lattice, this factor is always trivial.

In this work, we generalize their result to also include point group symmetry $G_{\text{pt}} \simeq G_s/G_{\text{trans}}$. In the following we will refer to $\tilde{G} = G \times G_{\text{pt}}$ as the extended onsite symmetry group. One can always choose the unit cell to be invariant geometrically under G_{pt} . Then the microscopic degrees of freedom in the unit cell should form a representation of \tilde{G} , classified again by $[\nu] \in \mathcal{H}^2[\tilde{G}, \text{U}(1)]$. We will show that Eq. (4) should continue to hold in this more general setting.

The derivation of Eq. (4) in Ref. [29] is based on the observation that the 2D lattice system can be viewed as surface of a three-dimensional weak SPT state, and then the constraint on symmetry fractionalization is obtained from bulk-boundary correspondence. Unfortunately, we do not know how to generalize this approach to include point-group symmetries, since the bulk-surface correspondence of 3D SPT states protected by point-group symmetries is not well-understood (however, see Ref. [47] for a recent development). Instead, here we give a purely two-dimensional argument, which not only reproduces the result of Ref. [29] for onsite symmetries, but also generalizes to point-group symmetries.

We consider a (simply-connected) region A that contains n unit cells (thus n background spinons), where n is an odd number. The ground state wave function can be Schmidt decomposed with respect to the cut ∂A :

$$|\Psi\rangle = \sum_{\alpha} \lambda_{\alpha} |\psi_A\rangle_{\alpha} \otimes |\psi_{\bar{A}}\rangle_{\alpha}. \quad (5)$$

Here \bar{A} is the complement of A . In the Schmidt decomposition, the Hilbert space \mathcal{H} is decomposed into $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$, where $\mathcal{H}_{A/\bar{A}}$ are supported on the interior and the exterior of the cut ∂A , respectively. $|\psi_A\rangle_{\alpha} \in \mathcal{H}_A$ and $|\psi_{\bar{A}}\rangle_{\alpha} \in \mathcal{H}_{\bar{A}}$ are Schmidt eigenstates in the two subspaces. Since $\{|\psi_A\rangle_{\alpha}\}$ form a complete orthonormal basis of \mathcal{H}_A , one can define the \tilde{G} -representation in the sub-

space \mathcal{H}_A using the basis of Schmidt eigenstates,

$$R_{\mathbf{g}}|\psi_A\rangle_\alpha = \sum_\beta U_{\alpha\beta}(\mathbf{g})|\psi_A\rangle_\beta, \quad \mathbf{g} \in \tilde{G}. \quad (6)$$

Here $R_{\mathbf{g}}$ should be understood as the restriction of the global \mathbf{g} transformation to the interior of A . The matrices $U(\mathbf{g})$ form a projective representation of \tilde{G} :

$$U(\mathbf{g})U(\mathbf{h}) = \omega_U(\mathbf{g}, \mathbf{h})U(\mathbf{gh}), \quad (7)$$

with a $U(1)$ phase ambiguity. Hence, different projective representations are classified by $\mathcal{H}^2[\tilde{G}, U(1)]$.

Our argument now roughly proceeds as follows: we will first analyze the symmetry action in a region in terms of the topological degrees of freedom, and then match with the microscopic description of the same symmetry action.

On one hand, we notice that the group \tilde{G} acts on the Hilbert space \mathcal{H}_A , to which the Schmidt eigenstates $|\psi_A\rangle_\alpha$ belong. The action of \tilde{G} on \mathcal{H}_A is determined by the symmetry transformation of the physical degrees of freedom, which is a tensor product of the microscopic constitute on each site. Correspondingly, the factor set ω_U is given by

$$[\omega_U] = ([\nu])^n. \quad (8)$$

On the other hand, the projective representation $U(\mathbf{g})$ also encodes symmetry actions on the anyonic excitations. For our purpose, it is of crucial importance to understand the interplay of translation symmetry with the extended symmetry group \tilde{G} , which naturally come in two forms: 1) Fractionalization of translation symmetries manifests as background anyon charges (i.e. background spinons in our case). \tilde{G} can act projectively on the background spinons. 2) The commutation relations between the translation symmetries and other symmetry transformations from \tilde{G} can fractionalize. This phenomena is dubbed “anyonic spin-orbit coupling (SOC)” in Ref. [29].

First, we study the symmetry action on the background anyon charges. Intuitively, the entire region A contains a total b^n anyon charge. Under the assumption of symmetry localization [9], $R_{\mathbf{g}}$ should act projectively on b^n . Denoting the local symmetry transformations by $U_{\mathbf{g}}^{b^n}$, they satisfy

$$U^{b^n}(\mathbf{g})U^{b^n}(\mathbf{h}) = [\omega_{b^n}(\mathbf{g}, \mathbf{h})]^n U^{b^n}(\mathbf{gh}). \quad (9)$$

Secondly, the symmetry action can change the Schmidt state in a more subtle way: in the presence of anyonic SOC, symmetry action in A can create an “anyonic flux density” running across A , which in turn creates anyon charge densities on the boundary of A , but does not affect the local density matrix inside A at all. This anyonic flux density creation contributes additional phase factors to $\omega_U(\mathbf{g}, \mathbf{h})$.

In general, the anyonic flux density created by a symmetry operation \mathbf{g} can be represented using a “vector”

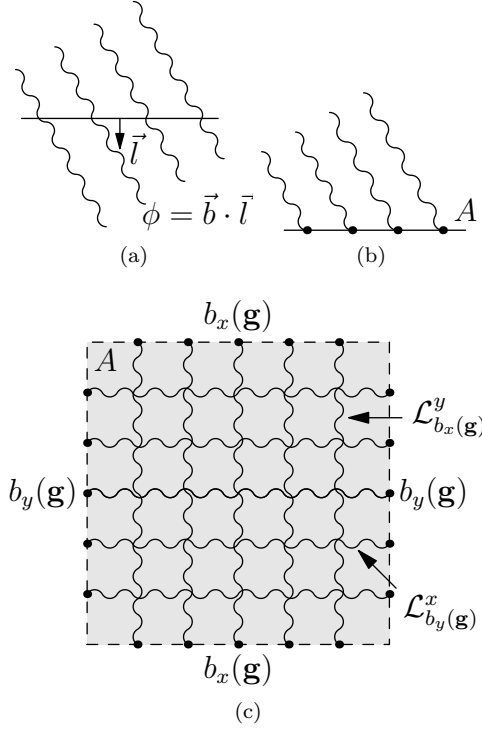


FIG. 1. Illustrations of the effect of anyon SOC.

$\vec{b}(\mathbf{g}) = b_x(\mathbf{g})\hat{x} + b_y(\mathbf{g})\hat{y}$. Here, the components of $\vec{b}(\mathbf{g})$ are Abelian anyon charges $b_{x,y}(\mathbf{g}) \in \mathcal{A}$. Such an anyon flux can be represented by a collection of anyonic string operators, as shown in Fig. 1(a). The total anyonic charge carried by these string operators going through \vec{l} is $\phi = \vec{b} \cdot \vec{l}$. In particular, these string operators terminate at the boundary of the region A , leaving an anyonic charge density, as shown in Fig. 1(b). In \mathbb{Z}_2 spin liquids, these background string operators can be thought as electric and magnetic field lines.

The value of the vector $\vec{b}(\mathbf{g})$ is determined by the fractionalization of commutation relations between \mathbf{g} and translational symmetries. We leave the computation of $\vec{b}(\mathbf{g})$ to Sec. IV, and examine now how the creation of an anyonic flux density $\vec{b}(\mathbf{g})$ contributes the extra phase factor τ to Eq. (4). We compute τ for the example of a rectangular region, with $n_{x,y}$ unit cells in x and y directions, respectively, as shown in Fig. 1(c). The symmetry action creates an n_x number of anyon string operators of the type $b_y(\mathbf{g})$ along the y direction, and an n_y number of $b_x(\mathbf{g})$ anyon strings along the x direction, respectively. Hence, the action of \mathbf{g} not only transforms the anyon within projectively, but also creates anyon string operators. Formally, we write this as

$$U(\mathbf{g}) = U^{b^n}(\mathbf{g})[\mathcal{L}_{b_x(\mathbf{g})}^x]^{n_y}[\mathcal{L}_{b_y(\mathbf{g})}^y]^{n_x}. \quad (10)$$

When combining the actions of \mathbf{g} and \mathbf{h} , the string operators contribute an additional phase factor from braid-

ing:

$$\begin{aligned} & [\mathcal{L}_{b_x(\mathbf{g})}^x]^{n_y} [\mathcal{L}_{b_y(\mathbf{g})}^y]^{n_x} [\mathcal{L}_{b_x(\mathbf{h})}^x]^{n_y} [\mathcal{L}_{b_y(\mathbf{h})}^y]^{n_x} \\ &= M_{b_y(\mathbf{g}), \overline{b_x(\mathbf{h})}}^{n_y n_x} [\mathcal{L}_{b_x(\mathbf{gh})}^x]^{n_y} [\mathcal{L}_{b_y(\mathbf{gh})}^y]^{n_x}. \end{aligned} \quad (11)$$

Combining this phase factor with the cocycle factors in Eqs. (8) and (9), we obtain

$$\omega_U(\mathbf{g}, \mathbf{h}) = [\nu(\mathbf{g}, \mathbf{h})]^n = [\omega_b(\mathbf{g}, \mathbf{h})]^n [M_{b_y(\mathbf{g}), \overline{b_x(\mathbf{h})}}]^n, \quad (12)$$

for general $n = n_x n_y$, and this in turn implies that

$$\nu(\mathbf{g}, \mathbf{h}) = \omega_b(\mathbf{g}, \mathbf{h}) M_{b_y(\mathbf{g}), \overline{b_x(\mathbf{h})}}. \quad (13)$$

This is the “anomaly-matching” constraint given at the beginning of this section in Eq. (4), where the twist factor $[\tau]$ has the following form,

$$\tau(\mathbf{g}, \mathbf{h}) = M_{b_y(\mathbf{g}), \overline{b_x(\mathbf{h})}}. \quad (14)$$

We notice that when \mathbf{g} and \mathbf{h} are onsite symmetries, this relation reproduces the twisted anomaly-matching condition, given by Eq. (52) of Ref. [29], up to a coboundary term. In fact, for onsite symmetries, $\vec{b}(\mathbf{g})$ is given by Eq. (19), and Eq. (13) can be written as

$$\begin{aligned} \nu(\mathbf{g}, \mathbf{h}) &= \omega_b(\mathbf{g}, \mathbf{h}) \frac{R^{\mathbf{b}(T_x, \mathbf{g}), \mathbf{b}(T_y, \mathbf{h})}}{R^{\mathbf{b}(T_y, \mathbf{g}), \mathbf{b}(T_x, \mathbf{h})}} \\ &\times \frac{R^{\mathbf{b}(T_y, \mathbf{gh}), \mathbf{b}(T_x, \mathbf{gh})}}{R^{\mathbf{b}(T_y, \mathbf{g}), \mathbf{b}(T_x, \mathbf{g})} R^{\mathbf{b}(T_y, \mathbf{h}), \mathbf{b}(T_x, \mathbf{h})}}, \end{aligned} \quad (15)$$

where the last term is a 2-coboundary. Hence, this anomaly-matching constraint is equivalent to the following,

$$\nu(\mathbf{g}, \mathbf{h}) = \omega_b(\mathbf{g}, \mathbf{h}) \frac{R^{\mathbf{b}(T_x, \mathbf{g}), \mathbf{b}(T_y, \mathbf{h})}}{R^{\mathbf{b}(T_y, \mathbf{g}), \mathbf{b}(T_x, \mathbf{h})}}, \quad (16)$$

which reproduces Eq. (52) of Ref. [29].

IV. ANYONIC FLUX DENSITY CREATED BY SYMMETRY OPERATIONS

In this section, we study the anyonic flux density created by a symmetry operation $\mathbf{g} \in \tilde{G}$. In particular, we compute the vector $\vec{b}(\mathbf{g})$ for both onsite and point-group symmetry operations, and show that the result can be related to the fractionalization of commutation relation between \mathbf{g} and translational symmetries, known as the anyonic SOC when \mathbf{g} is onsite. The results of this section is not only useful for deriving the extra phase factor $[\tau]$ in the constraint of Eq. (4), but also provides physical and observable effects of these symmetry fractionalization quantum numbers. Consequently, the creation of $\vec{b}(\mathbf{g})$ can be used as a way to define and to measure these symmetry fractionalization quantum number.

In the following we first review the relation between $\vec{b}(\mathbf{g})$ and anyonic SOC, for onsite symmetries, which is

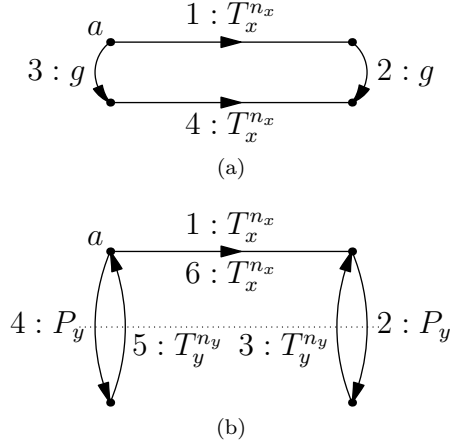


FIG. 2. Illustrations of the effect of anyon SOC.

discussed in details in Ref. [29], and then study the generalization to point-group symmetries.

First, we consider an onsite symmetry operation \mathbf{g} , which can be either unitary or anti-unitary (e.g. time-reversal symmetry). With anyonic SOC, \mathbf{g} action on the ground state effectively creates an anyonic flux density $\vec{b}(\mathbf{g})$. Such a flux density \vec{b} can be determined by moving a test anyon a along a vector $\vec{l} = l_x \hat{x} + l_y \hat{y}$. The commutation between the a string and the background string operators results in a phase $M_{a, \phi}$ where $\phi = \vec{b} \cdot \vec{l}$. Using this method, we will measure the anyonic flux density creation with the following thought experiment: consider creating a pair of anyons a and \bar{a} from the vacuum, and translate a by n_x unit lengths in the x -direction by applying a string operator \mathcal{L}_a^x . This can be thought as applying $T_x^{n_x}$ to a . We then apply the \mathbf{g} symmetry transformation to the state. The sequence is illustrated in Fig. 2(a) as $1 \rightarrow 2$. We can also switch the order of translating a and the \mathbf{g} action, corresponding to $3 \rightarrow 4$ in Fig. 2(a). Via the assumption of symmetry localization, it is easy to see that the Berry phase in this process is given by the commutator of $T_x^{n_x}$ and \mathbf{g} , and by definition is equal to $[\beta_a(T_x, \mathbf{g})]^{n_x}$. Using Eq. (2), this phase can be expressed as

$$[\beta_a(T_x, \mathbf{g})]^{n_x} = [M_{a, \mathbf{b}(T_x, \mathbf{g})}]^{n_x}. \quad (17)$$

Since the braiding phase $M_{a, c}$ is also the commutator of two crossing string operators moving a and c respectively, one can re-interpret Eq. (17) in the following manner: first, the \mathbf{g} action creates a string $\mathcal{L}_{b_y(\mathbf{g})}^y$ along y per unit length in the x -direction. The phase in Eq. (17) then results from the nontrivial commutator between \mathcal{L}_a^x and $\mathcal{L}_{b_x(\mathbf{g})}^x$ (the \mathcal{L}_a^x string have to cross n_x $\mathcal{L}_{b_y(\mathbf{g})}^y$ strings):

$$\mathcal{L}_a^x [\mathcal{L}_{b_x(\mathbf{g})}^y]^{n_x} = [M_{a, b_x(\mathbf{g})}]^{n_x} [\mathcal{L}_{b_x(\mathbf{g})}^y]^{n_x} \mathcal{L}_a^x. \quad (18)$$

This is illustrated in Fig. 2(a). Comparing the Berry phases in Eqs (17) and (18), we conclude that $b_y(\mathbf{g}) = \mathbf{b}(T_x, g)$. The other component $b_x(\mathbf{g})$ can be computed

similarly. Finally, we get the relation between $\vec{b}(\mathbf{g})$ and the anyonic SOC,

$$b_y(\mathbf{g}) = \mathbf{b}(T_x, \mathbf{g}), \quad b_x(\mathbf{g}) = \overline{\mathbf{b}(T_y, \mathbf{g})}. \quad (19)$$

Next, we show that point-group symmetries have a similar effect of creating anyonic flux densities associated with the fractionalization of their commutation relations with translations. Here, we only discuss mirror reflections, since other point-group symmetry operations can be reduced to them. Without losing generality, we describe our results in the context of mirror reflections on a square lattice. For simplicity, for the rest of this section we restrict ourselves to the case of a \mathbb{Z}_2 (toric code) topological order.

We first consider a mirror symmetry P_y , which maps y to $-y$. The anyonic flux created by P_y in the y direction, which is perpendicular to the mirror axis, can also be related to the fractionalization of the commutation relation between T_x and P_y . To detect this flux density, we draw a test string operator along the mirror axis. The Berry phase of exchanging this test string operator \mathcal{L}_a^x with the anyonic flux $[\mathcal{L}_{b_y(P_y)}^y]^{n_x}$, can be interpreted as the Berry phase of exchanging two sequences of symmetry actions shown in Fig. 2(b): The first sequence is applying L_a^x and then P_y , which maps to translating an a anyon by $T_x^{n_x}$ followed by applying P_y , and finally applying $T_y^{n_y}$ to move the anyon back to the end of the string operator. In Fig. 2(b) this sequence is $1 \rightarrow 2 \rightarrow 3$. The second sequence is applying P_y and then L_a^x , which maps to first applying P_y , then applying $T_y^{n_y}$ to move the anyon back to the beginning of the string operator, and finally apply $T_x^{n_x}$. This is illustrated as $4 \rightarrow 5 \rightarrow 6$ in Fig. 2(b). Here, additional translations in the y direction ($T_y^{n_y}$) are applied to ensure that the string operator \mathcal{L}_a^x appears at the same location in the two sequences. The Berry phase can be computed as the following,

$$\begin{aligned} T_x^{n_x} T_y^{n_y} P_y a &= [\omega_a(T_x, T_y)]^{n_x n_y} T_y^{n_y} T_x^{n_x} P_y a \\ &= [\omega_a(T_x, T_y)]^{n_x n_y} [\omega_a(T_x, P_y)]^{n_x} T_y^{n_y} P_y T_x^{n_x} a. \end{aligned} \quad (20)$$

The two interpretations of the same Berry phase should be equated:

$$M_{a, b_y(P_y)} = [\omega_a(T_x, T_y)]^{n_y} \omega_a(T_x, P_y). \quad (21)$$

Formally we should have

$$b_y(P_y) = \mathbf{b}(T_x, P_y) [\mathbf{b}(T_x, T_y)]^{n_y}. \quad (22)$$

When $\mathbf{b}(T_x, T_y) \neq \mathbb{1}$, this equation only gives consistent results if n_y has a fixed parity.

Let us go back to Eq. (21). To determine $b_y(P_y)$, we need at least two test anyons, which we will choose to be the background spinon $b = \mathbf{b}(T_x, T_y)$ and the vison v . For the background spinon, $\omega_b(T_x, T_y) = 1$ so the choice of n_y does not matter. If the test anyon is a vison, we now argue that the parity of n_y only depends on the

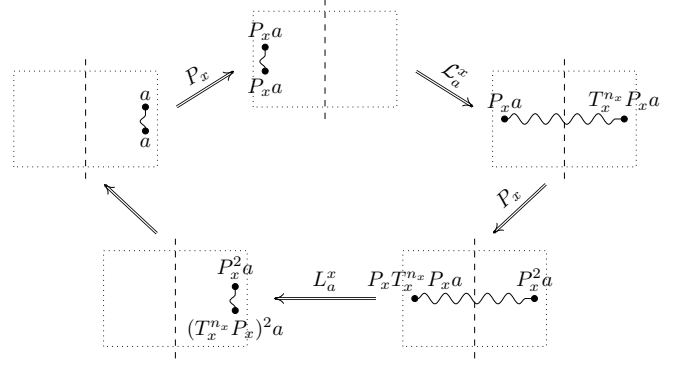


FIG. 3. Illustration of the physical process determining the anyonic flux creation $b_y(P_x)$.

geometric property of the mirror axis. For example, consider the site-centered reflection P_y . To see why, consider moving a vison by $T_y^{-n_y} T_x^{-n_x} T_y^{n_y} T_x^{n_x}$, so that the path is symmetric under P_y . $n_x n_y$ unit cells are enclosed by the path and the Berry phase is $(-1)^{n_x n_y}$. On the other hand, since the area enclosed by the path is symmetric under P_y , it is obvious that the number of physical spins inside the area must be $n_x \bmod 2$. We thus conclude that $(-1)^{n_x n_y} = (-1)^{n_x}$, which implies n_y is odd. Combining these arguments, we can unambiguously conclude that

$$b_y(P_y) = \mathbf{b}(T_x, P_y) \mathbf{b}(T_x, T_y). \quad (23)$$

We also notice that site-centered P_y is needed for our general proof to work, since there must be an odd number of spin-1/2's in the region A .

Similarly, $b_y(P_x)$ is related to the symmetry fractionalization associated with T_x and P_x . Because P_x and T_x does not commute, but instead satisfies a twisted commutation relation $T_x P_x = P_x T_x^{-1}$, we can not naively apply the previous definition of commutation relation fractionalization. As we show below, $b_y(P_x)$ is determined by a twisted version of commutation relation fractionalization,

$$b_y(P_x) = \mathbf{w}(T_x, P_x) \mathbf{w}(T_x, T_x^{-1}) \overline{\mathbf{w}(P_x, T_x^{-1})}. \quad (24)$$

Similar conclusions hold for P_y, T_y as well as the inversion I with $T_{x,y}$.

The anyonic flux density $b_y(P_x)$ can be computed from the Berry phase associated with applying a test string operator L_a^x and P_x in opposite orders. In particular, the Berry phase $M_{a, b_y(P_x)}$ is obtained through the following operations: $(\mathcal{L}_a^x)^{-1} P_x^{-1} \mathcal{L}_a^x P_x = \mathcal{L}_a^x P_x \mathcal{L}_a^x P_x$. This Berry phase can be related to symmetry fractionalization, by viewing these operations as symmetry actions on test anyon charges. As shown in Fig. 3, we first create two test charges of type a from the vacuum, to the right of the mirror axis, and denote the initial state by $a \times a$. The operation P_x reflects both anyons to the left, and we denote the result by $P_x a \times P_x a$. Next, the test string \mathcal{L}_a^x is created by translating one of the anyon to the right, and the state becomes $P_x a \times T_x P_x a$. Next,

P_x maps them to $P_x T_x P_x a \times P_x^2 a$. Finally, applying the test string operator moves the anyon on the left back to the right, and we get $T_x P_x T_x P_x a \times P_x^2 a$. Comparing this to the initial state, we conclude that the Berry phase accumulated is $\omega^a(T_x P_x, T_x P_x) \omega^a(P_x, P_x) = M_{a, b_y(P_x)}$. Hence, we conclude that

$$b_y(P_x) = \mathbf{w}(T_x P_x, T_x P_x) \mathbf{w}(P_x, P_x). \quad (25)$$

Using the cocycle equation $\mathbf{w}(\mathbf{g}, \mathbf{h}) \mathbf{w}(\mathbf{g}\mathbf{h}, \mathbf{k}) = \mathbf{w}(\mathbf{h}, \mathbf{k}) \mathbf{w}(\mathbf{g}, \mathbf{h}\mathbf{k})$, one can rewrite this result in the form of Eq. (24).

We notice that to determine the anyonic flux density created by a mirror symmetry operation, it is more convenient to decompose the flux density \vec{b} into components in directions parallel and perpendicular to the mirror axis, as in our discussion above, because in such a setup, the test string is invariant under the mirror symmetry. In the contrary, the flux density creation associated with onsite and inversion symmetries can be determined using decompositions in any directions.

Besides determining the twist factor $[\tau]$ in Eq. (14), the effect of anyonic flux density creation can also be used in defining and detecting the (twisted) commutation relation fractionalization between translations and elements of \tilde{G} . The Berry phase computations we have presented can serve as physical definitions of these fractionalization classes, and the anyonic flux creation provides an intuitive picture for the physics.

The effect of anyonic flux density creation can also be exploited in numerical diagnosis of the corresponding symmetry fractionalization classes. The fractionalization of commutation relations between one translation and another (unitary) symmetry operation \mathbf{g} can be measured as the difference in eigenvalues of \mathbf{g} operators on ground states in different topological sectors [18, 19], with appropriate lattice geometry. This result can be readily reproduced using anyonic flux density creation. For simplicity, we consider the example of an onsite unitary symmetry \mathbf{g} . The symmetry operation \mathbf{g} creates anyonic flux along the y direction, which can be represented as string operators $[\mathcal{L}_{b_y(\mathbf{g})}^y]^{n_x}$. On a minimally entangled state (MES) on a torus, with an anyonic flux a going through along the x direction, these string operators contribute a nontrivial phase factor $[M_{a, b_y(\mathbf{g})}]^{n_x} = [\beta_a(T_x, \mathbf{g})]^{n_x}$. Thus, if n_x is odd, the \mathbf{g} eigenvalues of different MESs encode the commutation relation fractionalization $\beta_a(T_x, \mathbf{g})$.

V. APPLICATION TO GAPPED SPIN LIQUIDS ON THE KAGOME/TRIANGULAR LATTICE

In this section, we apply the constraint Eq. (4) to gapped spin liquids in spin- $\frac{1}{2}$ models on the kagome/triangular lattice. This constraint greatly reduces the number of possible symmetry fractionalization classes on spinons. Together with previously obtained

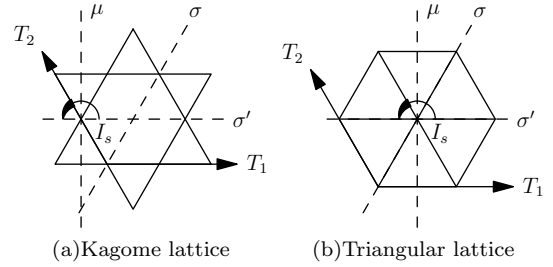


FIG. 4. Generators of the space group of the kagome/triangular lattice, $G_s = p6m$.

constraints on symmetry fractionalization of visons [26], we obtain a complete classification of gapped symmetric \mathbb{Z}_2 spin liquids.

To begin, we recall that the space group of the triangular/kagome lattice, $G_s = p6m$, is an extension of the translational symmetry group G_{trans} , generated by T_1 and T_2 , by the point group $G_{\text{pt}} = C_{6v}$, generated by two mirror reflections μ and σ , as shown in Fig. 4. It is also convenient to define a site-centered inversion I_s , as shown in Fig. 4, and the mirror reflection with the mirror axis along the T_1 direction $\sigma' = I_s \mu$. Here, we choose two mirror symmetries μ and σ' , whose axes intersect at a lattice site (consequently, the product of mirror reflections gives site-centered rotations), such that a region symmetric under two mirror reflections contain an even number of unit cells. We notice that the relation between σ' (and also I_s) and the generators of $p6m$ is different for the two different lattices, although in both cases we have defined $I_s = \mu\sigma'$.

To classify symmetry fractionalizations we first compute the second group cohomology $\mathcal{H}^2[\mathcal{G}, \mathbb{Z}_2] = \mathbb{Z}_2^8$. The $2^8 = 256$ different cohomology classes are labeled by eight \mathbb{Z}_2 invariants:

$$(\omega^{\text{spin}}, \omega^{12}, \omega^\mu, \omega^\sigma, \omega^{I_s}, \omega^T, \omega^{\mu T}, \omega^{\sigma T}), \quad (26)$$

where $\omega^{\text{spin}} = 0$ or $\frac{1}{2}$ denotes whether the anyon carries an integer or a half-integer spin, respectively; $\omega^{12} = \beta(T_1, T_2) = \pm 1$; the other six variables are all given in the form of $\omega^X = \omega(X, X) = \pm 1$, denoting the fractional quantum number $X^2 = \pm 1$. As we have already discussed in Sec. II, we need to specify $[\omega_e]$ and $[\omega_m]$.

The fact that $\omega_{\text{spin}}^e = 1/2$ and $\mathbf{b}(T_x, T_y) = e$ completely determines the symmetry fractionalization of the vison in a gapped \mathbb{Z}_2 spin liquid [26]. First, the background spinon charge density implies that the T_1 and T_2 anti-commute when acting on a vison, i.e. $\beta_m(T_1, T_2) = -1$. Intuitively, a vison acquires a Berry phase of π when it is adiabatically transported around a unit cell, because of the braiding with the background spinon charge. All the other fractional quantum numbers can be constrained using the method of flux-fusion anomaly test [25, 26]. To summarize, the symmetry fractionalization of visons is fixed:

$$[\omega_m] = (0, -1, +1, +1, -1, +1, +1, +1). \quad (27)$$

We notice that, for the convenience of applying the constraint in Eq. (4), we consider a site-centered inversion I_s , instead of a plaquette-centered inversion I_p as in Ref. [26]. As a result, the vison carries $I_s^2 = -1$. As discussed in Ref. [26], this is equivalent to $I_p^2 = +1$, and the vison PSG specified in Eq. (27) is the same as the odd Ising gauge theory on the kagome/triangular lattice [48–51]. We also notice that, although the fractional quantum numbers in Eq. (27) appear to be the same for vison PSGs on the kagome and triangular lattices, the two vison PSGs are indeed different [?]. This is because that these quantum numbers are defined in terms of symmetry operations I_s , which has different forms on the two types of lattices, in terms of the generators of the $p6m$ space group.

Now we turn to the symmetry fractionalization of spinons. To apply Eq. (4) we first study the symmetry representations of physical degrees of freedom in a unit cell. As discussed in Sec. III, they are classified by $\mathcal{H}^2[\tilde{G}, \text{U}(1)] = \mathbb{Z}_2^5$, labeled by the following invariants:

$$(\omega^{\text{spin}}, \omega^T, \omega^{\mu T}, \omega^{\sigma T}, \omega^{\mu, \sigma'}). \quad (28)$$

The first four invariants already appear in Eq. (26) [i.e. they are actually $\text{U}(1)$ invariants], and the last one $\omega^{\mu, \sigma'} = \beta(\mu, \sigma') = \pm 1$ represents the commutation relation fractionalization between μ and σ' . For spin-1/2 magnets on the triangular/kagome lattices, it is straightforward to check that the invariants are given by $[\omega_{\text{phys}}] = (\frac{1}{2}, -1, -1, -1, +1)$.

It is very useful to notice that there is a natural embedding of $\mathcal{H}^2[\tilde{G}, \text{U}(1)]$ into $\mathcal{H}^2[\tilde{G}, \mathbb{Z}_2]$. Namely, a nontrivial cohomology class of \tilde{G} with $\text{U}(1)$ coefficients is necessarily a nontrivial class with \mathbb{Z}_2 coefficients. More concretely, we already mentioned that $\omega^{\text{spin}}, \omega^T, \omega^{\mu T}, \omega^{\sigma T}$ are identified between the two, and $\omega^{\mu, \sigma'} = \omega^{I_s} \omega^\mu \omega^\sigma$ [This is because the symmetry fractionalization of $I_s^2 = (\mu\sigma')^2$ can be factorized as the product of μ^2 , σ'^2 , and the commutation relation between μ and σ']. ω^μ and ω^{I_s} are the only true “ \mathbb{Z}_2 ” invariants in $\mathcal{H}^2[\tilde{G}, \mathbb{Z}_2]$.

We now apply the constraint in Eq. (4) to the fractionalization of the background spinon b . First, we can show that the twist factor τ in Eq. (4) is always trivial. We observe that, among the invariants listed in Eq. (26) that label different fractionalization classes in $\mathcal{H}^2[\tilde{G}, \mathbb{Z}_2]$, none of them involves translations and \tilde{G} at the same time, i.e. no anyonic SOC appears in the list. In fact, as shown by an explicit computation in Appendix A, all anyonic SOC fractionalization classes $\mathbf{b}(T_{1,2}, \mathbf{g})$ (including the twisted version in Eq. (24)), and hence the anyonic flux densities $b_{x,y}(\mathbf{g})$, are either 1 or the background spinon $b = \mathbf{b}(T_1, T_2)$ for all $\mathbf{g} \in \tilde{G}$. Since in a \mathbb{Z}_2 spin liquid, we always have $M_{1,b} = M_{b,b} = +1$, the twist factor $[\tau]$ evaluates to 1 identically.

With $[\tau]$ out of the way, the constraint in Eq. (4) states that the symmetry fractionalization of the background spinon must match the projective representation of the physical degrees of freedom in each unit cell. In particular, it means that among the eight invariants,

$\omega_b^{\text{spin}}, \omega_b^T, \omega_b^{\mu T}, \omega_b^{\sigma T}$ are directly fixed by the symmetry properties of the unit cell, and ω_b^σ is not independent. In summary, we establish the following constraints:

$$\begin{aligned} \omega_b^{\text{spin}} &= \omega_{\text{phys}}^{\text{spin}}, \omega_b^\sigma = \omega_{\text{phys}}^{\mu, \sigma'} \omega_b^{I_s} \omega_b^\mu, \\ \omega_b^X &= \omega_{\text{phys}}^X, \quad X = T, \mu T, \sigma T. \end{aligned} \quad (29)$$

The remaining unconstrained ones are just $\omega_b^{12}, \omega_b^{\mu}$ and $\omega_b^{I_s}$, so we have $2^3 = 8$ classes of spinon PSGs. Since all the vison PSGs are completely fixed, we conclude that there are at most eight distinct types of symmetric \mathbb{Z}_2 spin liquid with a background spinon charge. There are actually no further constraints since all eight possible states have been constructed previously using bosonic [13] or fermionic parton constructions [14, 20, 52, 53], and tensor product states [15]. Therefore, we have obtained a complete classification of gapped symmetric \mathbb{Z}_2 spin liquids in translation-invariant spin- $\frac{1}{2}$ kagome/triangular lattice models. The results are summarized in Table I.

$[\omega]$	Relations	$[\omega_e]$	$[\omega_m]$
ω^{spin}	$(e^{i\pi S^x, y, z})^2 = \pm 1$	$\frac{1}{2}$	0
ω^{12}	$T_1 T_2 = \pm T_2 T_1$	± 1	-1
ω^σ	$\sigma^2 = \pm 1$	± 1	+1
ω^μ	$\mu^2 = \pm 1$	$\omega_e^\sigma \omega_e^{I_s}$	+1
ω^{I_s}	$I_s^2 = (\mu\sigma')^2 = \pm 1$	± 1	-1
ω^T	$T^2 = \pm 1$	-1	+1
$\omega^{\sigma T}$	$(\sigma T)^2 = \pm 1$	-1	+1
$\omega^{\mu T}$	$(\mu T)^2 = \pm 1$	-1	+1

TABLE I. Fractional quantum numbers in symmetric gapped \mathbb{Z}_2 spin liquids on the kagome lattice.

VI. CONCLUSION

In this work, we study the classification of symmetry fractionalization carried by spinons in gapped 2D \mathbb{Z}_2 spin liquids. For gapped spin liquids realized in spin systems with an odd number of spin- $\frac{1}{2}$ degrees of freedom per unit cell, we generalize the constraint of Cheng *et al.* [29] to point-group symmetries. The generalized constraint establishes a relation between the symmetry fractionalization of the spinon and the symmetry representation of the physical degrees of freedom in one unit cell. Applying this constraint to triangular/kagome lattices, we obtain a full classification (up to stacking SPT layers) of gapped \mathbb{Z}_2 spin liquids. In particular, we show that the eight spin liquid states constructed previously exhaust all possibilities [13–15, 20, 52, 53].

We notice that the Schwinger fermion construction actually yields 20 different PSGs for \mathbb{Z}_2 spin liquid states on the kagome lattice [14, 20]. 8 of them correspond to symmetric gapped spin liquids in Table I [20], while the rest, if naively translating the PSG to fractionalization class of

the fermionic spinons, violate the constraints in Eq. (29). However, the fermionic mean-field states are always gapless with these 12 classes of PSG, i.e. they are *gapless* \mathbb{Z}_2 spin liquids. Moreover, on other lattices, spin liquids where all anyons carry integer spins and/or $T^2 = +1$ [and thus violating the constraint in Eq. (4)] can also be realized as gapless spin liquids [11, 54, 55]. It was known that the gaplessness of fermionic spinons are protected by the PSG at the level of mean-field states, and our derivation of the constraints in Eq. (29) provides a non-perturbative understanding of the symmetry-protected gaplessness.

As we mentioned earlier, Ref. [29] argued that 2D systems with a spin-1/2 per unit cell can be viewed as the surface of a 3D weak SPT state protected by $\text{SO}(3)$ and translation symmetries, if only physical degrees of freedom carrying linear representations of $\text{SO}(3)$ are allowed. The surface of such a 3D SPT state must be anomalous in a precise manner that can be cancelled by “anomaly in-flow” to the bulk, and a symmetric surface state can either be gapless or gapped with an intrinsic topological order satisfying the constraint in Eq. (4). We believe such a bulk-surface correspondence can be generalized to point-group symmetries as well, and for gapped symmetry-preserving surface states Eq. (4) can be viewed as the precise form of such correspondence [29]. On the other hand, gapless \mathbb{Z}_2 spin liquids violating Eq. (29) found in Schwinger fermion constructions should also satisfy such a bulk-surface relation, i.e. they exhibit the correct anomaly. It will be interesting to investigate the relation further, which we will leave for future studies.

We notice that our constraints on the symmetry fractionalization of spinons can be easily adapted to spin models with only $\text{U}(1)_{S_z}$ spin-rotational symmetry (e.g. the $\text{SO}(3)$ symmetry is broken by easy-plane anisotropies), as long as there is still a background “spinon” charge per unit cell. Here a spinon should be understood as an anyon carrying 1/2 charge under the $\text{U}(1)_{S_z}$ symmetry. However, the constraints on the symmetry fractionalization of visons are much less stringent: the flux-fusion anomaly test only fixes ω_m^T, ω_m^μ and ω_m^σ [25], leaving other quantum numbers of visons undetermined. We leave a systematic classification of non-anomalous gapped spin liquids with $\text{U}(1)_{S_z}$ symmetry for future works.

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Appendix A: Anyonic spin-orbit coupling on the kagome lattice

In this appendix, we derive the symmetry fractionalization classes $\mathbf{b}(T_{1,2}, \mathbf{g})$, for symmetry operations $\mathbf{g} \in \tilde{G}$ on the kagome lattice. Because of the six-fold rotational symmetry, the second group cohomology $\mathcal{H}^2[G, \mathbb{Z}_2]$ factorizes as the following:

$$\mathcal{H}^2[\mathcal{G}, \mathbb{Z}_2] = \mathcal{H}^2[G_{\text{trans}}, \mathbb{Z}_2] \times \mathcal{H}^2[\tilde{G}, \mathbb{Z}_2], \quad (\text{A1})$$

where $\mathcal{H}^2[G_{\text{trans}}, \mathbb{Z}_2] = \mathbb{Z}_2$ is labeled by the quantum number $\omega^{12} = \beta(T_1, T_2)$. Therefore, there are no independent quantum numbers describing the anyon SOC, which must then be completely fixed by the value of ω^{12} .

We first show that an onsite symmetry operation \mathbf{g} must have a trivial anyonic SOC, i.e. $\beta(T_{1,2}, \mathbf{g}) = 1$. Using the cocycle condition of $\mathcal{H}^2[\mathcal{G}, \mathbb{Z}_2]$, one can show that

$$\beta(\mathbf{g}\mathbf{h}, \mathbf{k}) = \beta(\mathbf{g}, \mathbf{k})\beta(\mathbf{h}, \mathbf{k}), \quad (\text{A2})$$

if both \mathbf{g} and \mathbf{h} commute with \mathbf{k} . Since T_1 and T_2 are related by the C_6 rotational symmetry, the commutation-relation fractionalization between \mathbf{g} and $T_{1,2}$ must be the same: $\beta(T_1, \mathbf{g}) = \beta(T_2, \mathbf{g})$. Formally, this can be proved using Eq. (A2) and the relation $C_6^2 T_1 C_6^{-2} = T_2$. Then, using Eq. (A2) one more time, we get $\beta(T_1 T_2, \mathbf{g}) = \beta(T_1, \mathbf{g})\beta(T_2, \mathbf{g}) = +1$, i. e. \mathbf{g} must commute with $T_1 T_2$. However, the translational symmetry operation of $T_1 T_2$ is also related to both T_1 and T_2 by the C_6 symmetry. Therefore, we obtain $\beta(T_1, \mathbf{g}) = \beta(T_2, \mathbf{g}) = 1$. In particular, the anyonic SOC associated with the time-reversal symmetry, $\beta(T_{1,2}, T)$, must always be trivial.

Next, we study the symmetry fractionalization classes $\beta(T_{1,2}, \mathbf{g})$ when \mathbf{g} is a point-group symmetry operation. Instead of deriving them formally using the algebraic relations, here we compute them using an alternative approach, by explicitly constructing matrix representations of all $2^4 = 16$ cohomology classes in $\mathcal{H}^2[G_s, \mathbb{Z}_2]$. A matrix representation is a map $\phi : G \rightarrow \text{GL}(V)$ where V is a finite-dimensional complex vector space. Each projective representation gives a cocycle ω , as the group multiplication is only realized projectively,

$$\phi(\mathbf{g})\phi(\mathbf{h}) = \omega(\mathbf{g}, \mathbf{h})\phi(\mathbf{g}\mathbf{h}), \quad \omega(\mathbf{g}, \mathbf{h}) = \pm 1, \quad (\text{A3})$$

and ω satisfies the cocycle condition automatically, because matrix multiplications are associative. Therefore, each projective representation belongs to a certain class of $\omega \in \mathcal{H}^2[G, \mathbb{Z}_2]$. If we tensor product two projective representations $\phi = \phi_1 \otimes \phi_2$, the factor set of ϕ is the product of those of ϕ_1 and ϕ_2 . Therefore, we only need to construct projective representations realizing each of the four root classes in $\mathcal{H}^2[G_s, \mathbb{Z}_2] = \mathbb{Z}_2^4$, which has one of the four invariants $\omega^{12}, \omega^\mu, \omega^\sigma$ and ω^I being -1 and all the other three being $+1$.

As discussed in Sec. V, the cohomology classes with $\omega^{12} = 1$ can be realized by a representation with $\phi(T_1) =$

ω^μ	ω^σ	ω^I	$\phi(\mu)$	$\phi(\sigma)$
-1	+1	+1	$i\tau^2$	τ^3
+1	-1	+1	τ^3	$i\tau^2$
+1	+1	-1	τ^3	τ^1

TABLE II. Projective representations realizing root comology classes with $\omega^{12} = 1$. Here, τ^i denotes the Pauli matrices.

$\phi(T_2) = \mathbb{1}$. In particular, the three root classes can be realized using projective representations summarized in Table II. The last root class with $\omega^{12} = -1$ can be realized using a 16-dimensional projective representation. Here, we use $e(i, j)$, $i, j = 0, \dots, 3$ to denote the 16 basis vectors, and the projective representation is specified as follows

$$\begin{aligned}
\phi(T_1)e(i, j) &= e(i+1, j) \\
\phi(T_2)e(i, j) &= (-1)^i e(i, j+1) \\
\phi(\mu)e(i, j) &= (-1)^{j(j+1)/2} e(-i+j, j) \\
\phi(\sigma)e(i, j) &= (-1)^{ij} e(j, i).
\end{aligned} \tag{A4}$$

In these equations, i and j are defined mod 4, i. e. $e(i, j) = e(i+4, j) = e(i, j+4)$. This projective representation is obtained by rewriting the $(p_1, p_2, p_3) = (1, 0, 0)$ state in Ref. [13] as a matrix representation.

Using this projective representation, one can explicitly compute the fractionalization of commutation relations associated with the mirror symmetries μ and σ , for the cohomology class $\omega^{12} = -1$. First, we calculate the commutation relations between mirror reflections and trans-

lations along the mirror axes:

$$\beta(T_1 T_2, \sigma) = \beta(T_1, \sigma') = \beta(T_1 T_2^2, \mu) = -1. \tag{A5}$$

Since in the other root cohomology classes with $\omega^{12} = +1$ all these phase factors are equal to +1, we conclude that, $\beta(T_1 T_2, \sigma) = \beta(T_1, \sigma') = \beta(T_1 T_2^2, \mu) = \beta(T_1, T_2)$ holds for all cohomology classes. This implies the following results in $\mathcal{H}^2[\mathcal{G}, \mathcal{A}]$:

$$\mathbf{b}(T_1 T_2, \sigma) = \mathbf{b}(T_1, \sigma') = \mathbf{b}(T_1 T_2^2, \mu) = \mathbf{b}(T_1, T_2). \tag{A6}$$

Using Eq. (23), we see that no nontrivial anyonic flux densities are created by these mirror symmetries along the mirror axis.

Next, we consider the twisted commutation relation between mirror reflections and translations perpendicular to the mirror axes. Motivated by Eq. (24), we define the following quantum number,

$$\tilde{\beta}(\mathbf{g}, \mathbf{h}) = \omega(\mathbf{g}, \mathbf{h})\omega(\mathbf{g}, \mathbf{h}^{-1}\mathbf{g}\mathbf{h})/\omega(\mathbf{h}, \mathbf{h}^{-1}\mathbf{g}\mathbf{h}). \tag{A7}$$

It is easy to check that $\tilde{\beta}(\mathbf{g}, \mathbf{h})$ is a coboundary-independent invariant of a cohomology class $[\omega] \in \mathcal{H}^2[\mathcal{G}, \mathbb{Z}_2]$ for any two group elements. Using the explicit form of the projective representation in Eq. (A4), we can show that all the twisted commutation relations have trivial fractionalization,

$$\tilde{\beta}(T_1^{-1}T_2, \sigma) = \tilde{\beta}(T_1 T_2^2, \sigma') = \tilde{\beta}(T_1, \mu) = +1. \tag{A8}$$

Hence, for all cohomology classes in $\mathcal{H}^2[\mathcal{G}, \mathcal{A}]$, the results of Eq. (24) are trivial. In other words, the mirror symmetries do not create any anyonic flux density in the direction perpendicular to the mirror axes. Therefore, the mirror reflections μ , σ and σ' do not create any nontrivial anyonic flux density. As a result, we can safely ignore the $[\tau]$ factor, when applying the constraint in Eq. (4) to the triangular/kagome lattices in Sec. V.

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